ZT MAI 1 (14-15) - A

(1) $\lim\limits_{n\to\infty} n \lg\Big(1-\underbrace{\operatorname{tg}\frac{2}{n}}_{\to 0}\Big) = \infty\cdot 0^+ =$

$= \lim\limits_{n\to\infty} \dfrac{\lg\left(1-\operatorname{tg}\frac{2}{n}\right)}{\frac{1}{n}} = \lim\limits_{n\to\infty} \underbrace{\dfrac{\lg\left(1-\operatorname{tg}\frac{2}{n}\right)}{-\operatorname{tg}\frac{2}{n}}}_{\to 1} \cdot \underbrace{\dfrac{\operatorname{tg}\frac{2}{n}}{\frac{2}{n}}}_{\to 1} \cdot (-2)$ VLSF $\frac{2}{n}=t$

$\left(-\operatorname{tg}\frac{2}{n}=y\right)$ HV4 VLSF $= \lim\limits_{y\to 0} \underbrace{\dfrac{\lg(1+y)}{y}}_{\to 1} \cdot (-2) = -2$

(2) $\sum\limits_{1}^{\infty} (-1)^n \sin\left(\frac{n}{2+n^2}\right)$

a) abs. konvergence: $\sum\limits_{1}^{\infty} \sin\underbrace{\left(\frac{n}{2+n^2}\right)}_{\sim\frac{1}{n}}$ diverguje, neboť

$\lim \dfrac{\sin\left(\frac{n}{2+n^2}\right)}{\frac{1}{n}} = \lim\limits_{n\to\infty} \underbrace{\dfrac{\sin\frac{n}{2+n^2}}{\frac{n}{2+n^2}}}_{\to 1} \cdot \underbrace{\dfrac{\frac{n}{1+n^2}}{\frac{1}{n}}}_{\to 1} = 1,$

H.V. + VLSF $\lim\limits_{n\to\infty}\frac{n}{2+n^2}=0$

a $\sum \frac{1}{n}$ diverguje $\Rightarrow$ (limitní srovnávací kritérium) $\sum\limits_{1}^{\infty} \sin\left(\frac{n}{1+n^2}\right)$ diverguje.

b) neabsolutní konvergence: alternující řada $\left(0<\frac{n}{2+n^2}\le 1<\frac{\pi}{2} \Rightarrow\right.$
$\Rightarrow \sin\left(\frac{n}{2+n^2}\right)>0$

$\lim\limits_{n\to\infty} \sin\left(\frac{n}{2+n^2}\right) = \lim\limits_{x\to\infty} \sin\Big(\underbrace{\frac{x}{2+x^2}}_{\substack{\to 0\\ \neq 0}}\Big) \underset{\text{VLSF}}{=} \lim\limits_{u\to 0} \sin u = 0$

$\frac{n}{2+n^2} \searrow 0$ (např. $\left(\frac{x}{2+x^2}\right)' = \dfrac{2+x^2-x\cdot 2x}{(2+x^2)^2} = \dfrac{2-x^2}{(2+x^2)^2} < 0$

pro $x\ge 2$ (tj. pro $n\ge 2$)

$\Rightarrow$ funkce $f(x)=\frac{x}{2+x^2}$ a tedy i posl. členů $a_n=\frac{n}{2+n^2}$ je klesající

$\Rightarrow$ (Leibnizovo kritérium) $\sum (-1)^n \sin\left(\frac{n}{1+n^2}\right)$ je konvergentní,
tedy, daná řada konverguje neabsolutně

(3) $f(x) = \sqrt{\operatorname{arctg}(x-1)^2}$

a) $\operatorname{arctg}(x-1)^2 \geq 0 \Rightarrow f$ je def. v $\mathbb{R}$, f je spojitá a nezáporná!,

b) $f'(x) = \dfrac{1}{\cancel{2}\sqrt{\operatorname{arctg}(x-1)^2}} \cdot \dfrac{1}{1+(x-1)^4} \cdot \cancel{2}(x-1)$ pro $x \neq 1$

$$f'_{\pm}(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1\pm} \frac{\sqrt{\operatorname{arctg}(x-1)^2}}{x-1} = \text{„}\frac{0}{0}\text{"}$$

$$= \lim_{x\to 1\pm} \underbrace{\sqrt{\frac{\operatorname{arctg}(x-1)^2}{(x-1)^2}}}_{\to 1 \text{ (VLSF)}} \cdot \operatorname{sgn}(x-1) = \left\langle \begin{matrix} +1 \\ -1 \end{matrix} \right\} \Rightarrow$$

$$\lim_{t\to 0} \frac{\operatorname{arctg} t}{t} = 1$$

f nemá v $x=1$ derivaci oboustrannou, jen jednostranné!,
$f'_+(1) = 1$, $f'_-(1) = -1$

např: $\lim\limits_{x\to 1+} \dfrac{\sqrt{\operatorname{arctg}(x-1)^2}}{\underbrace{x-1}_{>0}} = \lim\limits_{x\to 1+} \dfrac{\sqrt{\operatorname{arctg}(x-1)^2}}{\sqrt{(x-1)^2}} = 1$

$\lim\limits_{x\to 1-} \dfrac{\sqrt{\operatorname{arctg}(x-1)^2}}{\underbrace{x-1}_{<0}} = \lim\limits_{x\to 1-} \dfrac{\sqrt{\operatorname{arctg}(x-1)^2}}{-\sqrt{(x-1)^2}} = \lim - \sqrt{\dfrac{\operatorname{arctg}(x-1)^2}{(x-1)^2}} = -1$

Stačí i v (jednostranných) limitách derivací

$$f'_{\pm}(1) = \lim_{x\to 1} \underbrace{\frac{x-1}{\sqrt{\operatorname{arctg}(x-1)^2}}}_{\text{„stejná" limita jako v definici}} \cdot \underbrace{\frac{1}{1+(x-1)^4}}_{\to 1}$$

(4) $f(x) = |x+2|\, e^{\frac{1}{x}}$

a) $D_f = \mathbb{R} - \{0\}$, $f(x) \geq 0$, $f(x) = 0 \iff x = -2$ – globální minimum fce

b) $\lim\limits_{x \to \pm\infty} |x+2|\, e^{\frac{1}{x}} = „\infty \cdot 1" = +\infty \Rightarrow$ f nenabývá globálního maxima

c) lokální extrémy:

$$f(x) = (x+2)\, e^{\frac{1}{x}} \cdot \operatorname{sgn}(x+2) \quad \text{v } \mathbb{R} - \{0\},\ x \neq -2$$

$$f'(x) = \left(e^{\frac{1}{x}} + (x+2)\, e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)\right) \operatorname{sgn}(x+2) =$$

$$= e^{\frac{1}{x}} \operatorname{sgn}(x+2)\left(1 - \frac{x+2}{x^2}\right) = e^{\frac{1}{x}} \operatorname{sgn}(x+2)\, \frac{x^2 - x - 2}{x^2} = \frac{e^{\frac{1}{x}}}{x^2}(x+1)(x-2) \cdot \operatorname{sgn}(x+2)$$

$$f'(x) = 0 \iff x^2 - x - 2 = 0 \iff (x-2)(x+2) = 0 \iff x = -1 \ \vee\ x = 2$$

f'	$-$	-2	$+$	-1	$-$	0	$-$	2	$+$
	$\searrow$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

v $x = -1$ f nabývá lok. maxima, $f(-1) = \frac{1}{e}$

v $x = 2$ f nabývá lok. minima, $f(2) = 4\sqrt{e}$ – není glob. min.

graf: ~~lim~~ f je spojitá v D_f,

$\lim\limits_{x \to 0-} \underbrace{|x+2|}_{\to 2}\, e^{\overbrace{\frac{1}{x}}^{\to -\infty}} = 0$, $\lim\limits_{x \to 0+} |x+2|\, \underbrace{e^{\frac{1}{x}}}_{\to\infty} = +\infty$

$$f'_+(-2) = \lim_{x \to -2+} f'(x) = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

$$f'_-(-2) = -e^{-\frac{1}{2}} = -\frac{1}{\sqrt{e}}$$

Konvexita grafu:

(Poznámka – vyšetření, kde je fce konvexní, resp. konkávní, inflexní body)

$$f''(x) = \left(e^{\frac{1}{x}} \left(1 - \frac{1}{x} - \frac{1}{x^2}\right) \operatorname{sgn}(x+2) \right)' \qquad x \neq -2$$

$$= \operatorname{sgn}(x+2)\, e^{\frac{1}{x}} \left[-\frac{1}{x^2}\left(1 - \frac{1}{x} - \frac{2}{x^2}\right) + \left(\frac{1}{x} + \frac{4}{x^3}\right) \right] =$$

$$= \operatorname{sgn}(x+2) \cdot e^{1/x} \cdot \frac{1}{x^4} \left(-x^2 + x + 2 + x^2 + 4x \right) =$$

$$= \operatorname{sgn}(x+2)\, e^{\frac{1}{x}} \cdot \frac{1}{x^4} (5x + 2) \quad , \; x \neq -2$$

$$f''(x) = 0 \Leftrightarrow x = -\frac{2}{5}$$

f'':	$+$		$-$		$+$		$+$
	$\cup$	-2	$\cap$	$-\frac{2}{5}$	$\cup$	0	$\cup$

v $x = -\frac{2}{5}$ je inflexe (navíc, $\lim_{x \to 0-} f'(x) = 0$)